NPS52-86-024

NAVAL POSTGRADUATE SCHOOL

Monterey, California

Deducing the Asymptotic Behavior of a Function from Its Behavior on a Subsequence

Daniel Davis

and

Hartmut Huber

November 1986

Approved for public release; distribution unlimited

Prepared for:

Chief of Naval Research
lington, VA 22217

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral R. C. Austin
Superintendent

D. A. Schrady
Provost

The work reported herein was supported in part by the Foundation Research Program of the Naval Postgraduate School with funds provided by the Chief of Naval Research and by a Contract from the Naval Weapons Center.

This report was prepared by:

VINCENT Y. LUM
Chairman
Department of Computer Science

KNEALE T. MARSHALL
Dean of Information and
Policy Science

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: NPS52-86-024
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): Deducing the Asymptotic Behavior of a Function from Its Behavior on a Subsequence
5. TYPE OF REPORT & PERIOD COVERED:
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): Daniel Davis, Hartmut Huber
8. CONTRACT OR GRANT NUMBER(s):
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Naval Postgraduate School, Monterey, CA 93943
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61152N: RR000-01, N0001486WR4E001
11. CONTROLLING OFFICE NAME AND ADDRESS: Chief of Naval Research, Arlington, VA 22217
12. REPORT DATE: November 1986
13. NUMBER OF PAGES: 19
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper investigates the relations between two complexity functions when they have the same order class on a subsequence. Sufficient conditions are derived for concluding that two functions have the same order class if they are sufficiently related on a subsequence. These theoretical results are used to prove order class properties of complexity functions arising from divide and conquer problems. In addition, it is shown that between any two comparable non-constant order classes there are order classes that are incomparable.

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S N 0102-LF-014-6601

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Deducing the Asymptotic Behavior of a Function from Its Behavior on a Subsequence

Daniel Davis and Hartmut Huber

Abstract

This paper investigates the relations between two complexity functions when they have the same order class on a subsequence. Sufficient conditions are derived for concluding that two functions have the same order class if they are sufficiently related on a subsequence. These theoretical results are used to prove order class properties of complexity functions arising from divide and conquer problems. In addition, it is shown that between any two comparable non-constant order classes there are order classes that are incomparable.

1. Introduction

The typical approach to determining the asymptotic complexity of a divide and conquer algorithm is to develop a recurrence relation on a subsequence of the domain of the complexity function, solve this recurrence, obtain the asymptotic complexity of the subsequence function and use this to deduce the asymptotic complexity of the function itself. In this paper, we closely examine the conditions that justify this last step. More precisely, under what conditions can we conclude that two complexity functions have the same order class, given that on some infinite subsequence, they have the same asymptotic complexity?

In section two we develop the primary results. In particular, we develop several sufficient conditions for concluding that two functions are comparable, given that they are comparable on a subsequence. If the order class of a function f is determined by the values of f on a sequence t then t is called a test sequence for f. Sufficient conditions are derived for defining coarser test sequences from a given test sequence. In the third section we examine some examples and apply our theoretical results to recurrence relations arising fron divide and conquer problems. In the fourth section we show that between any two comparable functions there are incomparable functions.

In the original work on this problem, we used standard methods. Gradually, it became evident that there was an inner structure underlying our results, and we found that our major results could be proved almost trivially using the concept of generalized inverses. The following is therefore of as much interest for the use of this concept as for the final results.

2. Primary Results

In the following, we will use the notation $\mathbf{N}$ for the nonegative integers, $\mathbf{R}$ for the real numbers. For relations and functions we use the following notation: $\subset$ for asymptotic dominance, $\ll$ for strict asymptotic dominance, $\equiv$ for asymptotic equivalence and $\mathbf{1}$ for the identity function on $\mathbf{N}$. Also we use the notation $s_k : \mathbf{N} \to \mathbf{N}$ for the "k - shift" function defined by $s_k(n) = n + k$. All functions are assumed to be monotonically increasing. When a function is strictly increasing, this will be stated where necessary. Throughout the following we will use the term "order class" to mean an equivalence class of the relation $\equiv$.

A sequence of nonnegative integers is described by a strictly increasing function $t: \mathbf{N} \to \mathbf{N}$. Such functions do not, in general, possess an inverse , but they always possess generalized inverses [1]. In fact, we have the following:

Lemma 2.1. If $t: \mathbf{N} \to \mathbf{N}$ is a strictly increasing function, then t has a unique generalized inverse $t^{\leftarrow}: \mathbf{N} \to \mathbf{N}$ such that:

$$t \circ t^{\leftarrow} \le 1 \quad \text{and} \quad t^{\leftarrow} \circ t = 1.$$

Proof: Define $t^{\leftarrow}$ by $t^{\leftarrow}(n) = m$ if and only if m is an integer such that $t(m) \le n < t(m+1)$. Such an m clearly exists. Then we have $t \circ t^{\leftarrow}(n) \le n < t \circ s_1 \circ t^{\leftarrow}(n)$. Also, it is easy to check that $t \circ t^{\leftarrow} \circ t = t$ and $t^{\leftarrow} \circ t \circ t^{\leftarrow} = t^{\leftarrow}$. If t is strictly increasing then m is unique, and if $n = t(m)$, then $t^{\leftarrow}(m) = n$. Thus $t^{\leftarrow}(t(n)) = n$.

Lemma 2.2. If $t: \mathbf{N} \to \mathbf{N}$ is a strictly increasing function, then t has a unique generalized inverse $t^{\rightarrow}: \mathbf{N} \to \mathbf{N}$ such that:

$$t \circ t^{\rightarrow} \ge 1 \quad \text{and} \quad t^{\rightarrow} \circ t = 1.$$

Proof: Define $t^{\rightarrow}$ by $t^{\rightarrow}(n) = m$ if and only if m is an integer such that $t(m-1) < n \le t(m)$. Such an m clearly exists. Then we have $t \circ s_{-1} \circ t^{\rightarrow}(n) < n \le t \circ t^{\rightarrow}(n)$. Also, it is easy to check that $t \circ t^{\rightarrow} \circ t = t$ and $t^{\rightarrow} \circ t \circ t^{\rightarrow} = t^{\rightarrow}$. If t is strictly increasing then m is unique, and if $n = t(m)$, then $t^{\rightarrow}(m) = n$. Thus $t^{\leftarrow}(t(n)) = n$ if t is strictly increasing.

Associated with each sequence t are two difference functions δ_t and Δ_t defined as follows:

Definition: If $t: \mathbf{N} \to \mathbf{N}$ is a strictly increasing function, define the *lower difference function* $\delta_t: \mathbf{N} \to \mathbf{N}$, and *upper difference function* $\Delta_t: \mathbf{N} \to \mathbf{N}$ by:

$$\delta_t = t \circ t^{\leftarrow} \quad \text{and} \quad \Delta_t = t \circ t^{\rightarrow}$$

Lemma 2.3 Given a strictly increasing function $t: \mathbf{N} \to \mathbf{N}$, the functions δ_t and Δ_t have the following properties:

1. $\delta_t \leq 1$ and $1 \leq \Delta_t$
2. $\delta_t \circ t = t$ and $\Delta_t \circ t = t$ (left neutral relative to t)
3. $\delta_t \circ \delta_t = \delta_t$ and $\Delta_t \circ \Delta_t = \Delta_t$ (idempotent)

Proof: Straightforward.
The following theorem is the first important result.

Theorem 2.1. Consider two functions $f, g: \mathbf{N} \to \mathbf{R}$, and assume that there exists a strictly increasing function $t: \mathbf{N} \to \mathbf{N}$ such that

$$f \circ \delta_t \equiv f$$

then

$$f \circ t \subset g \circ t \text{ iff } f \subset g.$$

<u>Proof</u>: The relation $\subset$ is right invariant under composition. Thus,

$$f \equiv f \circ \delta_t = f \circ t \circ t^{\leftarrow} \subset g \circ t \circ t^{\leftarrow} = g \circ \delta_t \leq g$$

Corollary 2.1.1. If $f, g: \mathbf{N} \to \mathbf{R}$ are two functions and $t: \mathbf{N} \to \mathbf{N}$ is a strictly increasing function, such that:

$$f \circ \delta_t \equiv f, \quad g \circ \delta_t \equiv g,$$

then

$$f \circ t \equiv g \circ t \text{ iff } f \equiv g.$$

Also, the result that $f \circ \delta_t \equiv f$ is a property of an order class follows from invariance of the relation $\equiv$ under right composition and the above theorem.

Corollary 2.1.2. - If $f, g: \mathbf{N} \to \mathbf{R}$ and $f \equiv g$, then

$$f \circ \delta_t \equiv f \text{ iff } g \circ \delta_t \equiv g.$$

Theorem 2.2. Consider two functions $f, g: \mathbf{N} \to \mathbf{R}$, and assume that there exists a strictly increasing function $t: \mathbf{N} \to \mathbf{N}$ such that

$$g \circ \Delta_t \equiv g$$

then

$$f \circ t \subset g \circ t \text{ iff } f \subset g.$$

Proof: The relation $\subset$ is right invariant under composition. Thus,

$$f \le f \circ \Delta_t = f \circ t \circ t^{\rightarrow} \subset g \circ t \circ t^{\rightarrow} = g \circ \Delta_t = g$$

Of course, we have analogous corollaries.

Corollary 2.2.1 If $f, g: \mathbb{N} \to \mathbb{R}$ are two functions and $t: \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, such that:

$$f \circ \Delta_t \equiv f, \quad g \circ \Delta_t \equiv g,$$

then

$$f \circ t \equiv g \circ t \quad \text{iff} \quad f \equiv g.$$

Also, the result that $g \circ \Delta_t \equiv g$ is a property of an order class follows from invariance of the relation $\equiv$ under right composition and the above theorem.

Corollary 2.2.2. - If $f, g: \mathbb{N} \to \mathbb{R}$ and $f \equiv g$, then

$$g \circ \Delta_t \equiv g \quad \text{iff} \quad f \circ \Delta_t \equiv f.$$

There are some additional properties of the generalized inverses of the functions that define test sequences that can be used to find additional, coarser test sequences.

Lemma 2.4. If $s,t : \mathbb{N} \to \mathbb{N}$ are strictly increasing, then

$$(s \circ t)^{\leftarrow} = t^{\leftarrow} \circ s^{\leftarrow} \text{ and } (s \circ t)^{\rightarrow} = t^{\rightarrow} \circ s^{\rightarrow}.$$

Proof: Both parts of the proof are similar so we do only one.
From $s^{\leftarrow} \circ s = 1$ and $t^{\leftarrow} \circ t = 1$, it follows that $t^{\leftarrow} \circ s^{\leftarrow} \circ s \circ t = 1$.
Similarly, from $s \circ s^{\leftarrow} \le 1$, and $t \circ t^{\leftarrow} \le 1$, it follows that

$$s \circ t \circ t^{\leftarrow} \circ s^{\leftarrow} \le s \circ s^{\leftarrow} \le 1.$$

Thus, $t^{\leftarrow} \circ s^{\leftarrow}$ has the unique defining properties of $(s \circ t)^{\leftarrow}$.

Note that the sequence defined by $u: \mathbb{N} \to \mathbb{N}$ is a subsequence of the sequence defined by $s: N \to N$, if there exists a sequence $t: \mathbb{N} \to \mathbb{N}$, such that $u = s \circ t$.

Theorem 2.3. Let $s, t: \mathbf{N} \to \mathbf{N}$, be strictly increasing and $f: \mathbf{N} \to \mathbf{R}$. If

$$f \circ \delta_s \equiv f \text{ and } (f \circ s) \circ \delta_t \equiv f \circ s,$$

then

$$f \circ \delta_{s \circ t} \equiv f.$$

Proof: $f \circ \delta_{s \circ t} = f \circ s \circ t \circ (s \circ t)^{\leftarrow} = f \circ s \circ t \circ t^{\leftarrow} \circ s^{\leftarrow} = f \circ s \circ s^{\leftarrow} \equiv f \circ \delta_s \equiv f.$

In essence this theorem states that if s is a test function for the order class of f and t is a test function for the order class defined by $g = f \circ s$, then $s \circ t$ is also a test function for f. Of course, $s \circ t$ is a subsequence of s and therefore is a "coarser" test sequence than s. An analogous result holds for the upper difference function Δ_t.

Theorem 2.4. Let $s, t: \mathbf{N} \to \mathbf{N}$, be strictly increasing and $f: \mathbf{N} \to \mathbf{R}$. If

$$f \circ \Delta_s \equiv f \text{ and } (f \circ s) \circ \Delta_t \equiv f \circ s,$$

then

$$f \circ \Delta_{s \circ t} \equiv f.$$

Proof: $f \circ \Delta_{s \circ t} = f \circ s \circ t \circ (s \circ t)^{\rightarrow} = f \circ s \circ t \circ t^{\rightarrow} \circ s^{\rightarrow} = f \circ s \circ s^{\rightarrow} \equiv f \circ \Delta_s \equiv f.$

3. Examples and Applications

We call the condition $f \circ \delta_t \equiv f$ the *lower growth* condition of f for t and $f \circ \Delta_t \equiv f$ the *upper growth* condition. Corollary 1.1.1 and Corollary 1.2.1 say that any two functions f,g that satisfy a growth condition for t and are equivalent on t have the same order class. Thus, if f satisfies a growth condition for the sequence t then the order class of f on t determines the order class of f uniquely.

Example 3.1: The growth conditions of f for t are the weakest conditions that allow us to conclude $f \subset g$ if $f \circ t \subset g \circ t$. Thus, if $f \circ \delta_t \not\equiv f$ then one can find a $g: N \to R$ such that $f \circ t \subset g \circ t$ but $f \not\subset g$. Obviously, $g = f \circ \delta_t$ is such a function since $g \circ t = f \circ t \circ t^{\leftarrow} \circ t = f \circ t \supset f \circ t$ but $g = f \circ \delta_t \not\supset f$.

Similarly, if $g \circ \Delta_t \not\equiv g$ then $f = g \circ \Delta_t$ satisfies $g \circ t = f \circ t \supset f \circ t$ but

$f = g \circ \Delta_t \not\subset g$.

Example 3.2: The condition $f \circ t \circ s_1 \equiv f \circ t$ is stronger than the growth condition of f for t and need not be satisfied even though $f \circ \delta_t \equiv f$, $f \circ t \subset g \circ t$ and hence $f \subset g$. For example, let $g: \mathbf{N} \to \mathbf{R}$, $t: \mathbf{N} \to \mathbf{N}$ be such that $g \circ t \circ s_1 \not\equiv g \circ t$ and define $f: \mathbf{N} \to \mathbf{N}$ by $f = g \circ \delta_t$. Then $f \circ \delta_t = g \circ \delta_t \circ \delta_t = g \circ \delta_t = f$, $f \circ t = g \circ t \subset g \circ t$ and $f \subset g$ but $f \circ t \circ s_1 \not\equiv f \circ t$.

Example 3.3: The two conditions $f \circ \delta_t \equiv f$ and $f \circ \Delta_t \equiv f$ are not equivalent. They are, in fact, independent conditions. Let $g: \mathbf{N} \to \mathbf{R}$ be such that $g \circ \delta_t \not\equiv g \circ \Delta_t$ and define $f_1, f_2: \mathbf{N} \to \mathbf{R}$ by $f_1 = g \circ \delta_t$, $f_2 = g \circ \Delta_t$. Then

$$f_1 \circ \delta_t \equiv f_1 \text{ but } f_1 \circ \Delta_t \not\equiv f_1 \text{ and}$$
$$f_2 \circ \Delta_t \equiv f_2 \text{ but } f_2 \circ \delta_t \not\equiv f_2.$$

3.1 Bounding functions for δ_t and Δ_t. It follows from the definitions of δ_t and Δ_t that the function $L_t = t \circ s_{-1} \circ t^{\rightarrow}$ is a lower bound for δ_t and that $U_t = t \circ s_1 \circ t^{\leftarrow}$ is an upper bound for Δ_t. Thus, δ_t and Δt can be bounded as follows:

$$L_t \le \delta_t \le 1 \le \Delta_t \le U_t$$

L_t and U_t are step functions that satisfy the conditions

$$L_t \circ U_t = \delta_t \qquad L_t \circ U_t \circ t = t$$
$$U_t \circ L_t = \Delta_t \qquad U_t \circ L_t \circ t = t$$

The following two lemmas establish the relationships between the growth conditions of f for t and similar conditions using L_t and U_t instead of δ_t and Δ_t.

Lemma 3.1. If $f: \mathbf{N} \to \mathbf{R}$ then $f \circ L_t \equiv f$ iff $f \circ U_t \equiv f$.

Proof: If $f \subset f \circ L_t$ then $f \circ U_t \subset f \circ L_t \circ U_t = f \circ \delta_t \subset f$, hence $f \circ U_t \equiv f$. Similarly, if $f \supset f \circ U_t$ then $f \circ L_t \supset f \circ U_t \circ L_t = f \circ \Delta_t \supset f$, hence $f \circ L_t \equiv f$.

Lemma 3.2. If $f \circ L_t \equiv f$ then $f \circ \delta_t \equiv f$ and $f \circ \Delta_t \equiv f$.

Proof: If $f \subset f \circ L_t$ then $f \circ \Delta_t \subset f \circ L_t \circ \Delta_t = f \circ L_t \subset f$, hence $f \circ \Delta_t \equiv f$. Also, if $f \subset f \circ L_t$ then $f \supset f \circ U_t$ and $f \circ \delta_t \supset f \circ U_t \circ \delta_t = f \circ U_t \supset f$, hence $f \circ \delta_t \equiv f$.

It should be noted, however, that the two conditions $f \circ \delta_t \equiv f$ together with $f \circ \Delta_t \equiv f$ are not equivalent to $f \circ L_t \equiv f$ (or $f \circ U_t \equiv f$). For example, if $t = 1$, then the function $f(n) = a^{2^{**}n}$ does not satisfy $f \circ L_t \equiv f$ since with $L_t = s_{-1}$, $f \circ L_t(n) = f(n-1) = a^{2^{**}(n-1)} \not\equiv f(n) = a^{2^{**}n} = (a^{2^{**}(n-1)})^2$. But $f \circ \delta t = f$ and $f \circ \Delta t = f$ hold. If, on the other hand, t is such that $t(n+1) > t(n) + 1$ for all $n \geq N_0$ for some N_0, then $f \circ \delta_t \supset f \supset f \circ \Delta_t$ implies $f \circ L_t \supset f \supset f \circ U_t$.

The values of L_t and U_t on the sequence t are defined by the equations:

$$L_t(t(n)) = t(n-1)$$
$$U_t(t(n)) = t(n+1).$$

Since $L_t \circ U_t = U_t \circ L_t = 1$ on $t(n)$, each of these equations is equivalent to the other one and represents the recurrence relation that defines the sequence $t(n)$. L_t expresses $t(n)$ as a function of $t(n+1)$, U_t expresses $t(n+1)$ as a function of $t(n)$, On the sequence t , L_t and U_t are inverses of each other.

Any monotonic function L such that $L \circ t = L_t \circ t$ is a lower bound for δ_t and any monotonic function U such that $U \circ t = U_t \circ t$ is an upper bound for Δ_t. Such bounding functions L, U can easily be computed for a sequence that is defined by a recurrence relation between $t(n)$ and $t(n+1)$.

For example, let t be defined by $t(0) = b$, $t(n+1) = bt(n)$ then $U = b1$ and $L = 1/b$. Or, to give a less trivial example, let t be defined by $t(0) = b$, $t(n+1) = t(n)^2$. In this case, $L(n) = \sqrt{n}$ and $U(n) = n^2$.

3.2 Divide and Conquer Problems: Our theoretical results can be used for proving order class properties of complexity functions arising from divide and conquer problems. For example, a result of two theorems (5.3.3 and 5.3.4) in [2] is that the order class of a monotonic function f that satisfies the recurrence relation:

$$f(1) = c$$
$$f(n) = af(n/b) + cn$$

on the sequence $t(k) = b^k$ is uniquely determined by the values of f on t. This result is proved by solving the recurrence relation, analyzing the solutions which have different forms depending on the relative size of the constants a, b, c, and establishing that in each case the growth of the solution is compatible with the growth of the sequence t to guarantee the unique order class of f. This is an unnecessarily complicated method for establishing a result that follows directly from the recurrence relation. Intuitively, one would expect this since the recurrence relation expresses directly the local growth property of f in relation to the growth of t that allows us to make conclusions about the order class of f. Our method applied to a more general problem is described below.

Theorem 3.2.1: Let $f, h, a : \mathbf{N} \to \mathbf{R}$, where a is bounded, and $g : \mathbf{N} \to \mathbf{N}$ be such that f satisfies

$$f(0) = c_0$$
$$f(n) = a(n)f(g(n)) + h(n)$$

on the sequence $n = t(k)$, $k \geq 0$ where $t(k)$ is defined by

$$t(0) = a_0$$
$$g(t(k)) = t(k-1).$$

If $h \circ t \subset h \circ t \circ s_{-1}$ then the order class of f is uniquely determined by the values of f on t.

Proof: Consider the recurrence relation for f on t:

$$\begin{aligned} f \circ t &= a * f \circ t \circ s_{-1} + h \circ t \\ &= a * a \circ s_{-1} \circ f \circ t \circ s_{-2} + a * h \circ t \circ s_{-1} + h \circ t \\ &\subset a * (a \circ s_{-1} \circ f \circ t \circ s_{-2} + h \circ t \circ s_{-1}) \\ &= a * f \circ t \circ s_{-1} \subset f \circ t \circ s_{-1} \end{aligned}$$

Hence $f \subset f \circ \Delta_t = f \circ t \circ t^{\rightarrow} \subset f \circ t \circ s_{-1} \circ t^{\rightarrow} = f \circ L_t \subset f \circ \delta_t$. By Corollary 2.2.1 the condition $f \equiv f \circ \delta_t$ determines the order class of f uniquely.

For example, $h(n) = cn$, $a(n) = a$, and $g(n) = n/b$ define the classical case of a complexity function arising from a divide and conquer problem with a linear overhead. Here $t(k) = b^k$ and the condition $h \circ t \subset h \circ t \circ s_{-1}$ holds since $b^k \subset b^{k-1} = b^k/b$.

More generally, the condition $h \circ t \subset h \circ t \circ s_{-1}$ is satisfied for the same sequence $t = b^k$ if h is any polynomial in n. However, if $g(n) = \sqrt{n}$, defining the sequence $t(n) = b^{2^{**}n}$, the condition $h \circ t \subset h \circ t \circ s_{-1}$ does not hold for a linear or polynomial function $h(n)$. It holds for $h(n) = $ constant. In fact, since for $h(n) = $ constant the condition $h \circ t \subset h \circ t \circ s_{-1}$ holds for any t, it is clear that in this case the values of f on any arbitrary sequence determine the order class of f uniquely.

4. Incomparable Order Classes

In this section we construct examples that demonstrate that, in general, no relationship concerning the order classes of two monotonic functions is determined by the fact that the functions coincide on an infinite subset of their domain. This is trivial if the functions are not required to be monotonic but it is not obvious for monotonic functions. As in the previous sections, all functions are assumed to be monotonically increasing. For the construction of the examples the following lemma and its corollary are needed.

Lemma 4.1: For any f, g: $\mathbb{N} \to \mathbb{R}$ the following statements are equivalent:

(1) $g \not\subset f$

(2) there exists a sequence S of integers such that for every infinite subsequence $\{n_i \mid n_i \in S, i = 0,1,...\}$ the sequence $g(n_i)/f(n_i)$ is strictly increasing and unbounded.

Proof:

1. Assume that $g \not\subset f$. Then there exists a sequence S_1 such that the sequence $g(m_i)/f(m_i)$ is unbounded for m_i in $S_1 = (m_1, m_2, ...)$. Select from S_1 a subsequence $S = (n_1, n_2, ...)$ such that for each n_i in S

$$g(n_{i+1})/f(n_{i+1}) \geq g(n_i)/f(n_i) + r$$

for a fixed real number r. This can always be achieved since the original sequence $g(m_i)/f(m_i)$ was unbounded. In addition, every infinite subsequence S' of S has the property that the sequence

$$g(k_i)/f(k_i), \quad k_i \text{ in } S'$$

is strictly increasing and unbounded.

2. Conversely, assume that $S = (n_1, n_2, ...)$ is a sequence of integers such that for $i = 1,2,...$

$$g(n_i)/f(n_i), \quad n_i \text{ in } S,$$

is unbounded. Then, a fortiori, for no constant K in R, the relation $g(n) < Kf(n)$ can hold for all but finite many values of n. Hence $g \not\subset f$.

Corollary 4.1: For any f,g: $\mathbf{N} \to \mathbf{R}$, let S be a sequence of integers such that $g(n_i)/f(n_i)$ is strictly increasing and unbounded for every subsequnce $(n_1, n_2, \ldots, n_i, \ldots)$ of S and let h: $\mathbf{N} \to \mathbf{R}$. Then,

$h \not\subset f$ if $h(n) = g(n)$ for an infinite subsequence of S and
$h \not\supset g$ if $h(n) = f(n)$ for an infinite subsequence of S.

Construction of the examples:

Let f,g: $\mathbf{N} \to \mathbf{R}$, $f \ll g$, and $f \gg$ const. We will first construct a function h such that $f \ll h \ll g$. This can be accomplished by splicing pieces of f and g and constant functions together. From h one can then construct two functions h_1 and h_2 that are not comparable. This situation is characterized by the condition: $h_1 \not\supset h_2$ and $h_1 \not\subset h_2$. The details of this construction are shown below.

Since $f \ll g$ there exists an integer N_f and a real number K such that $f(n) < Kg(n)$ for all $n \geq N_f$. Also, since $g \not\subset f$ there exists a sequence S of integers as described in lemma 4.1.

(a) Define the following sequence of integers N_j, $j \geq 0$:

$N_0 = N_f$

For $i \geq 0$:

N_{3i+1} = smallest integer k: $k > N_{3i}$ and k in S
N_{3i+2} = smallest integer k: $k > N_{3i+1}$ and $f(k) > Kg(N_{3i+1})$
N_{3i+3} = smallest integer k: $k > N_{3i+2}$ and k in S

(b) Define h:$N \to R$ as follows:

$h(n) = f(n)$	for $n \leq N_0$
$h(n) = Kg(n)$	for $N_{3i} < n \leq N_{3i+1}$
$h(n) = Kg(N_{3i+1})$	for $N_{3i+1} < n < N_{3i+2}$
$h(n) = f(n)$	for $N_{3i+2} \leq n \leq N_{3i+3}$

By construction, it follows that
(1) $f(n) \leq h(n) \leq Kg(n)$, hence $f \subset h \subset g$.
(2) $h(n) = Kg(n)$ on an infinite subset of S, hence $h \not\subset f$.
(3) $h(n) = f(n)$ on an infinite subset of S, hence $h \not\supset g$.
Therefore, $f \ll h \ll g$.

(c) Define $h_1, h_2 : \mathbf{N} \to \mathbf{R}$ as follows:

$h_1(n) = f(n)$ for $N_{6i} < n < N_{6i+2}$,
$h_1(n) = h(n)$ elsewhere

$h_2(n) = f(n)$ for $N_{6i+3} < n < N_{6i+5}$
$h_2(n) = h(n)$ elsewhere.

Since $h_1(n) = Kg(n)$ for all values of n from an infinite subset of S, it follows from the corollary that $h_1 \not\subset f$. From $f \subset h_2$ we obtain $h_1 \not\subset h_2$. Similarly, since $h_1(n) = f(n)$ for all values of n from an infinite subset of S, it follows from the corollary that $g \not\subset h_1$. From $h_2 \subset g$ we obtain $h_2 \not\subset h_1$. Therefore, none of the order class relations «, », ≡ hold between h_1 and h_2. These two functions are not comparable.

Thus, the concept "order class" does not characterize the asymptotic behavior of monotonic functions very well. It only classifies functions according to one single growth characteristic represented by the growth of the bounding function, bounding from above. The examples show that the growth of functions can flip between the growth characteristics of different order classes. Thus, while the order class concept provides a means for equivalencing monotonic functions and therefore partitioning the set of monotonic functions, it does not provide a means for ordering the resulting classes.

5. Conclusion

The results of the last section confirm the intuition that asymptotic complexity can be a poor means for comparing algorithms. In general, there is no asymptotic limit that reveals the true growth of an algorithm. Indeed, functions can behave more erratically in their limit than on any finite interval. Similarly, the notion that the asymptotic behavior of a function on an infinite subset of the domain should allow us to infer asymptotic behavior is false. Combining these results with the fact that, in general, we are interested in the behavior of algorithms on problems within a range of sizes, not in asymptotic behavior, we see the need for better, more useful methods for analyzing and comparing algorithms.

For some important classes of problems, however, we know that the complexity functions are sufficiently well behaved that their asymptotic growth reflects the true growth of the corresponding algorithms. The growth conditions developed above characterize precisely the meaning of the intuitive idea of a function being well behaved. In addition, these conditions provide a simple, yet powerful proof technique for inferring the overall growth behavior of a function from its growth behavior on a subset of the domain.

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